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# ORTHOGONAL LATIN SQUARES WITH SUBSQUARES

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Denote by LS$(v, n)$ a pair of orthogonal latin squares of side $v$ with orthogonal subsquares of side $n$. It is proved by using a generalized singular direct product that for every odd integer $n \geqslant 304$ or every even integer $n \geqslant 304$ in some infinite families, an LS$(v, n)$ exists if and only if $v \geqslant 3n$. It is also proved that for every integer $n \geqslant 304$, an LS$(v, n)$ exists if $v > 3n+6$.

## 1. Introduction

A *latin square* of side $v$ is a $v \times v$ array each of whose rows and columns is a permutation of a $v$-set $S$. Two latin squares of side $v$ are *orthogonal* if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. The following theorem completely settles the existence question for a pair of orthogonal latin squares.

**Theorem 1.1** (Bose, Shrikhande, Parker [1]). *A pair of orthogonal latin squares of side $v$ exists if and only if $v \notin \{2, 6\}$.*

Denote by LS$(v, n)$ a pair of orthogonal latin squares of side $v$ which have the remaining orthogonal subsquares of side $n$ if $v$-$n$ rows and $v$-$n$ columns are deleted. Since the cse $v = n$ is trivial, we suppose in this paper that $v > n$. The existence question of LS$(v, n)$ has been investigated by some authors. The following necessary condition is easy to prove, see Parker [10].

**Theorem 1.2.** *An* LS$(v, n)$ *exists only if $v \geqslant 3n$.*

For the sufficient condition. Horton [8] proved that an LS$(v, n)$ exists if $n \notin \{2, 6\}$ and $v$ is sufficiently large. Crampin and Hilton [3] gave a bound and proved that an LS$(v, n)$ exists if

$$v \geqslant 36373 \cdot 5^2 \cdot 2^{16} \cdot f(n)n^2,$$

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where $f(n)$ is a certain function with positive integral values. Recently, much progress has been made by Drake and Lenz [5].

**Theorem 1.3.** *An* LS$(v, n)$ *exists if* $v \geqslant 4n+3$ *and* $n \geqslant 304$.

Wallis and the author [11] have proved a necessary and sufficient condition for some small $n$.

**Theorem 1.4.** *For* $n = 3, 4, 5$, *an* LS$(v, n)$ *exists if and only if* $v \geqslant 3n$.

In this paper it is proved by using a generalized singular direct product that for every odd integer $n \geqslant 304$ or every even integer $n \geqslant 304$ in some infinite families, an LS$(v, n)$ exists if and only if $v \geqslant 3n$. It is also proved that for every integer $n \geqslant 304$, an LS$(v, n)$ exists if $v > 3n+6$.

## 2. Generalized singular direct product

First, we need the concept of an incomplete array (see Mullin [9]).

**Definition 2.1.** Let $N_i$ be an $n$-subset of a $v$-set $V_i$, $i = 1$ or 2. A $v \times v$ array $D$ is called an *incomplete array*, denoted by IA$(v, n)$ if:

(i) $D$ contains an $n \times n$ subarray $E$ with all its cells empty.

(ii) Any row or column which does not meet $E$ is a permutation of $V_i$ in the $i$th position, $i = 1$ or 2.

(iii) Any row or column which meets $E$ is a permutation of $V_i \backslash N_i$ in the $i$th position, $i = 1$ or 2.

(iv) Each member of $(V_1 \times V_2) \backslash (N_1 \times N_2)$ occurs in some cell of $D$.

The following is an example of an IA(20, 6) based on $V_1 = V_2 = \{0, 1, \ldots, 9, A$ $B, C, D\} \cup N_1$ and $N_1 = N_2 = \{P, Q, W, X, Y, Z\}$. The array $D$, a pair of squares with a $6 \times 6$ subarray $E$ missing in the bottom right corner is shown in Fig. 1. It will be used in Section 4.

Obviously, an IA$(v, 0)$ is also a pair of orthogonal latin squares of side $v$. It is easy to get an IA$(v, n)$ from an LS$(v, n)$ by deleting the orthogonal subsquares of side $n$. Then from Theorem 1.1, an IA$(v, 1)$ exists if $v \notin \{2, 6\}$. For $n = 2$, Heinrich [6] shows

**Theorem 2.2.** *An* IA$(v, 2)$ *exists if and only if* $v \geqslant 6$.

We now describe the generalized singular direct product of orthogonal latin squares, which is essentially Proposition 21 in Brouwer [2]. (We omit its proof.) Let $N(v)$ denote the maximum number of pairwise orthogonal latin squares of side $v$.

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2 | Y | W | X | P | 1 | D | 4 | Q | 9 | 8 | A | Z | 6 | B | 5 | C | 7 | 3 |
| Z | 1 | 3 | Y | W | X | P | 2 | 0 | 5 | Q | A | 9 | B | 7 | C | 6 | D | 8 | 4 |
| C | Z | 2 | 4 | Y | W | X | P | 3 | 1 | 6 | Q | B | A | 8 | D | 7 | 0 | 9 | 5 |
| B | D | Z | 3 | 5 | Y | W | X | P | 4 | 2 | 7 | Q | C | 9 | 0 | 8 | 1 | A | 6 |
| D | C | 0 | Z | 4 | 6 | Y | W | X | P | 5 | 3 | 8 | Q | A | 1 | 9 | 2 | B | 7 |
| Q | 0 | D | 1 | Z | 5 | 7 | Y | W | X | P | 6 | 4 | 9 | B | 2 | A | 3 | C | 8 |
| A | Q | 1 | 0 | 2 | Z | 6 | 8 | Y | W | X | P | 7 | 5 | C | 3 | B | 4 | D | 9 |
| 6 | B | Q | 2 | 1 | 3 | Z | 7 | 9 | Y | W | X | P | 8 | D | 4 | C | 5 | 0 | A |
| 9 | 7 | C | Q | 3 | 2 | 4 | Z | 8 | A | Y | W | X | P | 0 | 5 | D | 6 | 1 | B |
| P | A | 8 | D | Q | 4 | 3 | 5 | Z | 9 | B | Y | W | X | 1 | 6 | 0 | 7 | 2 | C |
| X | P | B | 9 | 0 | Q | 5 | 4 | 6 | Z | A | C | Y | W | 2 | 7 | 1 | 8 | 3 | D |
| W | X | P | C | A | 1 | Q | 6 | 5 | 7 | Z | B | D | Y | 3 | 8 | 2 | 9 | 4 | 0 |
| Y | W | X | P | D | B | 2 | Q | 7 | 6 | 8 | Z | C | 0 | 4 | 9 | 3 | A | 5 | 1 |
| 1 | Y | W | X | P | 0 | C | 3 | Q | 8 | 7 | 9 | Z | D | 5 | A | 4 | B | 6 | 2 |
| 7 | 8 | 9 | A | B | C | D | 0 | 1 | 2 | 3 | 4 | 5 | 6 | | | | | | |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | 0 | 1 | 2 | | | | | | |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | 0 | 1 | | | | | | |
| 8 | 9 | A | B | C | D | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | | | | | |
| 5 | 6 | 7 | 8 | 9 | A | B | C | D | 0 | 1 | 2 | 3 | 4 | | | | | | |
| 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | 0 | 1 | 2 | 3 | | | | | | |

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | Y | C | B | D | 9 | P | 6 | Q | A | W | X | Z | 1 | 7 | 2 | 3 | 8 | 4 | 5 |
| 2 | 1 | Y | D | C | 0 | A | P | 7 | Q | B | W | X | Z | 8 | 3 | 4 | 9 | 5 | 6 |
| Z | 3 | 2 | Y | 0 | D | 1 | B | P | 8 | Q | C | W | X | 9 | 4 | 5 | A | 6 | 7 |
| X | Z | 4 | 3 | Y | 1 | 0 | 2 | C | P | 9 | Q | D | W | A | 5 | 6 | B | 7 | 8 |
| W | X | Z | 5 | 4 | Y | 2 | 1 | 3 | D | P | A | Q | 0 | B | 6 | 7 | C | 8 | 9 |
| 1 | W | X | Z | 6 | 5 | Y | 3 | 2 | 4 | 0 | P | B | Q | C | 7 | 8 | D | 9 | A |
| Q | 2 | W | X | Z | 7 | 6 | Y | 4 | 3 | 5 | 1 | P | C | D | 8 | 9 | 0 | A | B |
| D | Q | 3 | W | X | Z | 8 | 7 | Y | 5 | 4 | 6 | 2 | P | 0 | 9 | A | 1 | B | C |
| P | 0 | Q | 4 | W | X | Z | 9 | 8 | Y | 6 | 5 | 7 | 3 | 1 | A | B | 2 | C | D |
| 4 | P | 1 | Q | 5 | W | X | Z | A | 9 | Y | 7 | 6 | 8 | 2 | B | C | 3 | D | 0 |
| 9 | 5 | P | 2 | Q | 6 | W | X | Z | B | A | Y | 8 | 7 | 3 | C | D | 4 | 0 | 1 |
| 8 | A | 6 | P | 3 | Q | 7 | W | X | Z | C | B | Y | 9 | 4 | D | 0 | 5 | 1 | 2 |
| A | 9 | B | 7 | P | 4 | Q | 8 | W | X | Z | D | C | Y | 5 | 0 | 1 | 6 | 2 | 3 |
| Y | B | A | C | 8 | P | 5 | Q | 9 | W | X | Z | 0 | D | 6 | 1 | 2 | 7 | 3 | 4 |
| 6 | 7 | 8 | 9 | A | B | C | D | 0 | 1 | 2 | 3 | 4 | 5 | | | | | | |
| 7 | 8 | 9 | A | B | C | D | 0 | 1 | 2 | 3 | 4 | 5 | 6 | | | | | | |
| 5 | 6 | 7 | 8 | 9 | A | B | C | D | 0 | 1 | 2 | 3 | 4 | | | | | | |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | 0 | 1 | 2 | | | | | | |
| B | C | D | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | | | | | | |
| C | D | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | | | | | | |

Fig. 1

**Theorem 2.3** (Generalized singular direct product). *Suppose* $N(q) \geq 3$, *and an* IA$(m+l_i, l_i)$ *exists where* $l_i \geq 0$, $1 \leq i \leq q$, $l_1 + l_2 + \cdots + l_q = k$. *Then an* IA$(qm+k, k)$ *exists. Moreover, if an* IA$(k, 0)$ *exists, then*

(i) *An* LS$(qm+k, k)$ *exists.*

(ii) *An* LS$(qm+k, m)$ *exists if there is at least one i such that* $l_i = 0$.

(iii) *An* LS$(qm+k, q)$ *exists if* $m > 2l_i$ *for all i,* $1 \leq i \leq q$.

For the case $l_1 \neq 0$, $l_2 = l_3 = \cdots = l_q = 0$, this construction becomes the following singular direct product (see [4, pp. 428–432]).

**Theorem 2.4** (Singular direct product). *Suppose $N(q) \geqslant 3$, and an $\mathrm{IA}(m, 0)$ and an $\mathrm{IA}(m+k, k)$ exist. Then an $\mathrm{IA}(qm+k, k)$ and an $\mathrm{IA}(qm+k, m+k)$ exist. Moreover, if an $\mathrm{IA}(m+k, 0)$ (or an $\mathrm{IA}(k, 0)$) exists, then*

(i) *An $\mathrm{LS}(qm+k, m+k)$ and an $\mathrm{LS}(qm+k, m)$ (also an $\mathrm{LS}(qm+k, k)$) exist.*

(ii) *An $\mathrm{LS}(qm+k, q)$ exists if $m > 2k$.*

Notice that Theorem 2.4 contains the existence of an $\mathrm{LS}(qm+k, m+k)$ which is useful in the following discussion, but Theorem 2.3 does not contain such a case. If $l_1 = l_2 = \cdots = l_q = 0$ in Theorem 2.3 we get the direct product. Let $\mathrm{LS}(m)$ denote a pair of orthogonal latin squares of side $m$.

**Theorem 2.5** (Direct product). *If an $\mathrm{LS}(m_1)$ and an $\mathrm{LS}(m_2)$ exist, then an $\mathrm{LS}(m_1 m_2)$ exists.*

The following result (see Wang and Wilson [12]) is helpful in using the generalized singular direct product.

**Theorem 2.6.** $N(v) \geqslant 3$ *if* $v \notin \{2, 3, 6, 10, 14\}$.

## 3. A bound

In this section one of the main results of this paper is proved, namely the following theorem.

**Theorem 3.1.** *For every integer $n \geqslant 304$, an $\mathrm{LS}(v, n)$ exists if $v > 3n+6$.*

For this we need two lemmas.

**Lemma 3.2.** *Suppose $n \geqslant 7$ and $N(n) \geqslant 3$. Then an $\mathrm{LS}(3n+i, n)$ exists for every $i$, $7 \leqslant i \leqslant n$.*

**Proof.** In Theorem 2.3, let $q = n$, $m = 3$, $l_1 = \cdots = l_i = 1$, $l_{i+1} = \cdots = l_q = 0$, $k = l_1 + \cdots + l_q = i$. Since an $\mathrm{IA}(4, 1)$ and an $\mathrm{IA}(3, 0)$ exist, then an $\mathrm{IA}(3n+i, i)$ exists. Since $7 \leqslant i \leqslant n$, from Theorem 1.1 and $\mathrm{IA}(i, 0)$ exists. Moreover, the condition (iii) in Theorem 2.3 is satisfied: $m > 2l_t$ for every $t$, $1 \leqslant t \leqslant q$. Then an $\mathrm{LS}(3n+i, n)$ exists.

**Lemma 3.3.** *Suppose $n \geqslant 5$ and $N(n) \geqslant 3$. Then an $\mathrm{LS}(4n+i, n)$ exists for $i = 1$ or 2.*

**Proof.** Let $q = n$, $m = 4$, $l_1 = 1$, $l_2 = \cdots = l_q = 0$. It is easy to see by using Theorem 2.3 similarly to the proof of Lemma 3.2 that an $\mathrm{LS}(4n+1, n)$ exists. For

$i=2$, let $q=4$, $m=n$ and $k=2$ in Theorem 2.4. From Theorem 2.6, $N(4)\geqslant 3$. Since $N(n)\geqslant 3$, then an $\mathrm{IA}(n,0)$ exists. Since $n\geqslant 5$, from Theorem 2.2 an $\mathrm{IA}(n+2,2)$ exists. Then we know from Theorem 2.4 that an $\mathrm{IA}(4n+2,\ n+2)$ exists. Moreover, an $\mathrm{IA}(n+2,0)$ exists since $n\geqslant 5$, then from Theorem 2.4(i) we know that an $\mathrm{LS}(4n+2,n)$ exists.

**Proof of Theorem 3.1.** Suppose $n\geqslant 304$, then from Theorem 2.6 $N(n)\geqslant 3$. The condition $v>3n+6$ can be divided into three subcases: (1) $v\geqslant 4n+3$, (2) $v=4n+i$, $i=1$ or 2, (3) $v=3n+i$, $7\leqslant i\leqslant n$. Then the conclusion follows from Theorem 1.3, Lemma 3.3 and Lemma 3.2 respectively.

In fact, we can further get the following lemma by using Theorem 2.3.

**Lemma 3.4.** *Suppose $N(n)\geqslant 3$. Then an $\mathrm{LS}(3n+i,n)$ exists for $i=0$, 1, 3, 4, 5.*

## 4. Necessary and sufficient condition

For some $n\geqslant 304$, the bound shown in Section 3 can be improved. We lower the above bound in this section for the case $n$ odd and the case $n$ even belonging to some infinite families of integers to prove that the necessary condition "$v\geqslant 3n$" is also sufficient. From Heinrich [7, Theorem 2], we have

**Lemma 4.1.** *Suppose $n$ is odd, $n\neq 3$. Then an $\mathrm{LS}(3n+2,n)$ exists.*

**Lemma 4.2.** *Suppose $n$ is odd, $n\geqslant 27$. Then an $\mathrm{LS}(3n+6,n)$ exists.*

**Proof.** Let $n=2r-1$, $r\geqslant 14$. Then

$$3n+6=2(n+3)+n=4(r+1)+2r-1.$$

In Theorem 2.3, let $q=r+1$, $m=4$, $l_1=l_2=l_3=1$, $l_4=\cdots=l_q=2$. Then an $\mathrm{IA}(3n+6,n)$ exists. Since an $\mathrm{IA}(n,0)$ exists, then an $\mathrm{LS}(3n+6,n)$ exists.

**Theorem 4.3.** *For every odd integer $n\geqslant 304$, an $\mathrm{LS}(v,n)$ exists if and only if $v\geqslant 3n$.*

**Proof.** The necessity comes from Theorem 1.2. The sufficiency can be obtained easily from Theorem 3.1, Lemma 4.2, Lemma 3.4 and Lemma 4.1.

We now discuss the case that $n$ is even. It is easy to see from the proof of Theorem 4.3 that we need only to consider the existence of $\mathrm{LS}(3n+6,n)$ and $\mathrm{LS}(3n+2,n)$.

**Definition 4.4.** An odd prime number $p$ is called *regular* if an $\mathrm{IA}(3p-1, p-1)$ exists.

**Lemma 4.5.** *Odd prime numbers* 3, 5 *and* 7 *are regular.*

**Proof.** The conclusion comes from Theorem 2.2, Theorem 1.4 and Fig. 1.

**Lemma 4.6.** *Suppose $n$ is even, $n \geqslant 12$, and $n+3$ is a composite number with at least one regular odd prime divisor. Then an $\mathrm{LS}(3n+6, n)$ exists.*

**Proof.** Let $n+3=pr$ and $p$ be the least regular odd prime divisor, then $r>1$. Since $n \geqslant 12$, $r \geqslant 5$ and then $N(r) \geqslant 3$. Write

$$3n+6=2(n+3)+n=r(2p)+(pr-3).$$

In Theorem 2.3, let $q=r$, $m=2p$, $l_1=l_2=l_3=p-1$, $l_4=\cdots=l_q=p$. Since $p$ is a regular odd prime number, an $\mathrm{IA}(3p-1, p-1)$ exists. From Theorem 2.5, an $\mathrm{IA}(3p, p)$ exists. Then we know from Theorem 2.3 that an $\mathrm{LS}(3n+6, n)$ exists.

**Lemma 4.7.** *Suppose $n$ is even, $n \geqslant 14$, and $n+1$ has at least one regular odd prime divisor. Then an $\mathrm{LS}(3n+2, n)$ exists.*

**Proof.** Let $n+1=pr$ and $p$ be the least regular odd prime divisor. If $r=1$, the conclusion is already true. Otherwise, since $n \geqslant 14$, then $r \geqslant 5$ and $N(r) \geqslant 3$. Write

$$3n+2=2(n+1)+n=r(2p)+(pr-1).$$

Let $q=r$, $m=2p$, $l_1=p-1$, $l_2=\cdots=l_q=p$. Similarly to the proof of Lemma 4.6, we know from Theorem 2.3 that an $\mathrm{LS}(3n+2, n)$ exists.

**Lemma 4.8.** *Suppose $n$ is even, $n \geqslant 304$, $n+3$ is a composite number, and both $n+3$ and $n+1$ have a regular odd prime divisor. Then an $\mathrm{LS}(v, n)$ exists if and only if $v \geqslant 3n$.*

**Proof.** The necessity is obvious. The sufficiency comes from Theorem 3.1, Lemma 4.6, Lemma 3.4 and Lemma 4.7.

**Theorem 4.9.** *Suppose $n$ belongs to one of the following infinite families of integers $S_i$,*

$$S_1=\{105l+15t+2 \mid l \geqslant 3, 0 \leqslant t \leqslant 6\},$$
$$S_2=\{105l+15t+9 \mid l \geqslant 3, 0 \leqslant t \leqslant 6\},$$
$$S_3=\{105l+21t+6 \mid l \geqslant 3, 0 \leqslant t \leqslant 4\},$$
$$S_4=\{105l+21t+11 \mid l \geqslant 3, 0 \leqslant t \leqslant 4\},$$
$$S_5=\{105l+35t+4 \mid l \geqslant 3, 0 \leqslant t \leqslant 2\},$$
$$S_6=\{105l+35t+27 \mid l \geqslant 3, 0 \leqslant t \leqslant 2\}.$$

*Then an $\mathrm{LS}(v, n)$ exists if and only if $v \geqslant 3n$.*

**Proof.** If $n$ is in some $S_i$, $n \geqslant 304$. If $n$ is odd, the conclusion is already true from Theoem 4.3. If $n$ is even, it is easy to check that $n+1$ and $n+3$ are composite numbers, each of them has a regular odd prime divisor belonging to the set $\{3, 5, 7\}$. Then the conclusion comes from Lemma 4.8.

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